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Some Issues Concerning Holographic Dark Energy

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Abstract

We study perturbation of holographic dark energy and find it be stable. We study the fate of the universe when interacting holographic dark energy is present, and discuss a simple phenomenological classification of the interacting holographic dark energy models. We also discuss the cosmic coincidence problem in the context of holographic dark energy. We find that the coincidence problem can not be completely solved by adding an interacting term. Inflation may provide a better solution of the coincidence problem.

1 Introduction

The cosmological constant problem[1] is a longstanding problem in theoretical physics, and has been taken more and more seriously since the discovery of accelerated expansion of our universe[2][3][4]. In addition to the problem why the cosmological constant is nonvanishing, there is also the "cosmic coincidence problem"[1].

Based on the validity of effective quantum field theory, Cohen *et al* [5] pointed out that the quantum zero-point energy of a system should not exceed the mass of a black hole of the same size. This observation relates the UV cutoff of a system to its IR cutoff. As a cosmological application, Li [6] suggested to choose the future event horizon as the IR cut-off, the energy density of vacuum is given by

$$\rho_D = 3c^2 M_p^2 R_h^{-2} \ , \tag{1}$$

where $R_h \equiv a \int_t^\infty dt'/a(t')$ is the size of future event horizon. This is called holographic dark energy (HDE) and has been studied extensively [7].

As a phenomenological model, the stability of holographic dark energy is an important issue and has been investigated by Myung [8] first. Myung [8] assumed that holographic dark energy is a usual fluid component. He calculated the square of sound speed of holographic dark energy and found it be negative, this leads to an instability of perturbation of holographic dark energy. However, holographic dark energy is given by the holographic vacuum energy, whose perturbation should be treated globally. A calculation of perturbation of holographic dark energy will be presented in sect.2. It is shown that perturbation of holographic dark energy is stable, and we do not need to face the negative sound speed square problem.

The interacting holographic dark energy, assuming that dark energy interacts with matter, has become a popular topic recently [9]. With the interacting term, the story of holographic dark energy becomes more interesting. There is often an attractor solution to the evolution equation, in which the effective equations of state of dark energy and matter become identical in the far future. In sect.3, we will give a simple and phenomenological classification of the interacting terms, we will show that we can tune the interacting parameter to avoid the phantom-like universe.

In sect.4, we shall discuss the coincidence problem of holographic dark energy in a more natural way. We end this paper with conclusion and discussion in sect.5.

2 Stability of Holographic Dark Energy

In this section, we investigate perturbation of holographic dark energy. First, we shall calculate perturbation of the future event horizon, consequently we get the density perturbation of holographic dark energy. Finally, we couple this density perturbation to gravity. As an application, we analyze the coupled equation approximately in the dark energy dominated era, and show that the perturbation is stable. We also solve the perturbation equation numerically outside the horizon. When dark energy dominates, the solution agrees with the analytic result. When matter dominates, the numerical result also shows that the perturbation is stable.

We consider the scalar type perturbation of the metric. In the Newtonian gauge, the perturbated metric takes the form

$$ds^2 = -(1 + 2\Phi(r,t))dt^2 + a(t)^2(1 - 2\Phi(r,t))d\mathbf{x}^2 \ , \tag{2}$$

where for simplicity, we have assumed that the perturbation is spherically symmetric, $\Phi = \Phi(r,t)$, where $r = |\mathbf{x}|$. In this metric, light traveling from the horizon towards the origin $r = 0$ still goes straightly. As illustrated in Fig. 1, the future event horizon R_h can be written as

$$R_h(0,t) = \int_0^{r_h(t)} a(t)(1 - \Phi(r,t))dr \ , \tag{3}$$

where $R_h(0,t)$ denotes the future event horizon at position $r = 0$ at time t, $r_h(t)$ denotes the coordinate distance of the future event horizon. At the first order in the perturbation theory, $r_h \equiv r_{h0} + \delta r_h$, where $r_{h0}(t) = \int_t^\infty dt'/a(t')$, and δr_h can be written as

$$\delta r_h(t) = \int_t^\infty \frac{2\Phi(r_{h0}(t'),t')}{a(t')} dt' \ . \tag{4}$$

So the variation of the future event horizon R_h at the position $r = 0$ takes the form

$$\delta R_h(0,t) \equiv R_h(0,t) - R_{h0} = a(t)\left\{\int_t^\infty \frac{2\Phi(r_{h0}(t'),t')}{a(t')} dt' - \int_0^{r_{h0}} \Phi(r,t)dr\right\} \ . \tag{5}$$

Note that for the background value, we have $R_{h0} = ar_{h0}$.

Using the definition of holographic dark energy (1), and varying R_h, we get the variation of the energy density of holographic dark energy with respect to the metric perturbation

$$\delta\rho_D = -2\rho_D \frac{\delta R_h}{R_h} \ . \tag{6}$$

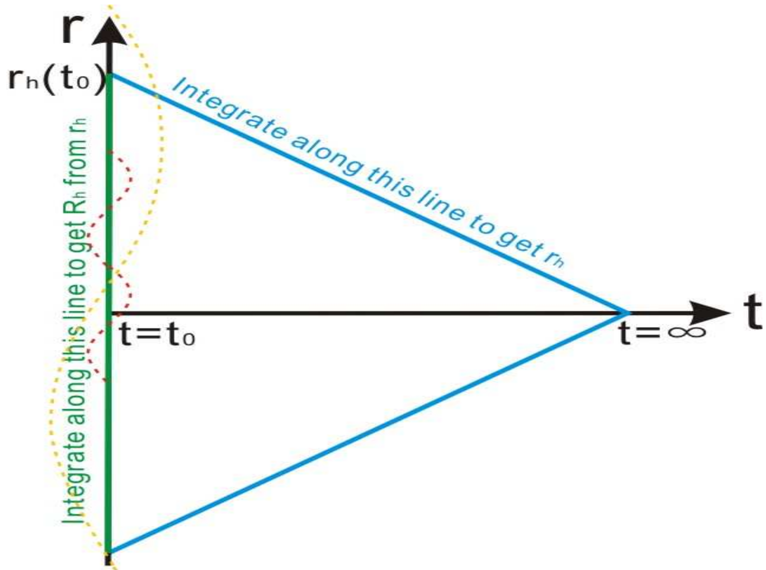

Figure 1: This figure illustrates how to calculate the perturbation of the future event horizon. We first integrate along the blue line, which is the geodesic for photons, to get the coordinate distance of the future event horizon r_h, and then integrate along the green line from $r = 0$ to $r = r_h$ to get the physical distance R_h. The dashed red and yellow curves illustrates the sub-horizon and super-horizon perturbations, respectively.

Inserting this equation into the 00-component of the perturbated Einstein equation, one obtains

$$\frac{\nabla^2}{a^2}\Phi - 3H\dot{\Phi} - 3H^2\Phi = \frac{1}{2M_p^2}(\delta\rho_D + \delta\rho_m) \ . \tag{7}$$

For simplicity, we neglect the matter density perturbation, so $\delta\rho_m = 0$. To solve this equation, we expand Φ using its eigenfunction. Write

$$\Phi(r,t) = \sum_k \Phi_k(t)\frac{\sin(kr)}{r} \ , \tag{8}$$

where we have dropped the $\cos(kr)/r$ terms, which lead to a singularity at $r = 0$. Then $\Phi_k(t)$ satisfies

$$\begin{aligned} &\frac{M_p^2}{\rho_D}\frac{\sin(kr)}{r}r_{h0}(t)\left\{\frac{k^2}{a^2}\Phi_k(t) + 3H\dot{\Phi}_k(t) + 3H^2\Phi_k(t)\right\} \\ = \ &\int_t^\infty \frac{2\Phi_k(t')\sin(kr_{h0}(t'))dt'}{a(t')r_{h0}(t')} - \Phi_k(t)\int_0^{r_{h0}(t)}\frac{\sin(kr)}{r}dr \ . \end{aligned} \tag{9}$$

One way to deal with this equation is to take derivative with respect to t. This

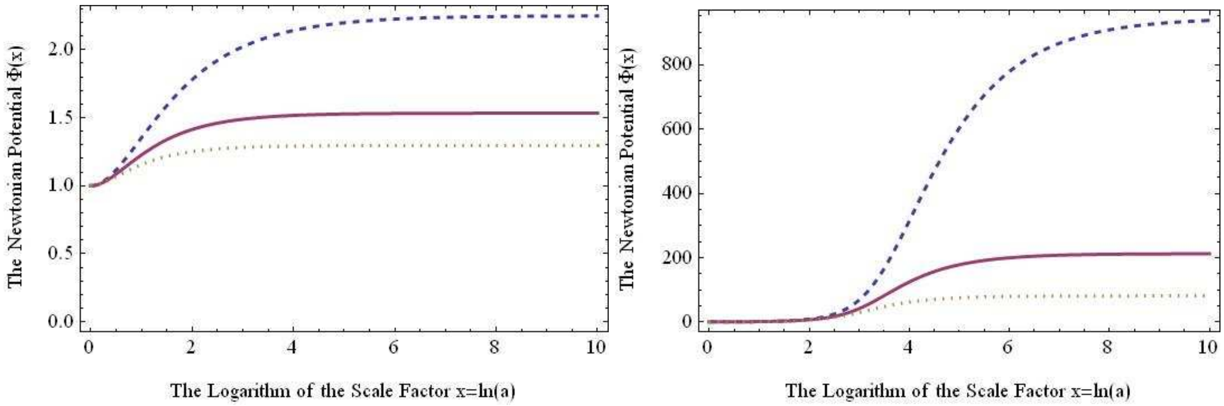

Figure 2: We plot the gravitational potential Φ as a function of $x \equiv \ln a$ in the $kr_{h0} \ll 1$ case. The left figure is plotted starting from the dark energy dominated era $\Omega_D|_{x=0} = 0.72$, and the right figure is plotted starting from the matter dominated era $\Omega_D|_{x=0} = 0.01$. The blue (dashed), red (solid) and yellow (dotted) curves corresponds to $c = 0.8$, $c = 1.0$ and $c = 1.2$ respectively. In both figures, we see that the perturbation approaches a constant mode. Note that we have chosen the initial condition $\Phi(0) = 0$ and $\Phi'(0) = 1$. So although in the right figure, the amplitude can grow $\mathcal{O}(100)$ times before approaching the constant mode, but as the initial condition should be set to $\Phi(0) \propto \rho_D$, the physical amplitude is still not too large.

integral equation becomes a differential equation

$$\begin{aligned} \ddot{\Phi}_k &+ \frac{1}{3H}\left\{\left(3\dot{H} + 9H^2 - \frac{9H}{R_{h0}} + \frac{k^2}{a^2}\right) + \frac{a\rho_D}{M_p^2 R_{h0}}\int_0^{r_{h0}} \frac{\sin(kr)}{kr}dr\right\}\dot{\Phi}_k \\ &+ \frac{1}{3H}\left\{\left(6H\dot{H} + 6H^3 - \frac{3}{R_{h0}}\frac{k^2}{a^2} - \frac{9H^2}{R_{h0}}\right) + \frac{\rho_D}{M_p^2 R_{h0}}\frac{\sin(kr_{h0})}{kr_{h0}}\right\}\Phi_k = 0 \ . \end{aligned} \quad (10)$$

This equation can be solved at least numerically. As an example, the evolution of the $kr_{h0} \ll 1$ mode is shown in Fig. 2.

Eq.(9) can also be treated directly in the dark energy dominated era $\rho_D > \rho_m$. To investigate the stability of the perturbation mode, we focus on behavior of $\dot{\Phi}_k/\Phi_k$. When $\dot{\Phi}_k/\Phi_k \to 0$, the perturbation mode is frozen, and when $\dot{\Phi}_k/\Phi_k < 0$, the perturbation mode is decaying.

Let us first consider the super-horizon mode $kr_{h0}(t) \ll 1$. In this case, Eq. (9) can be written as

$$3H\dot{\Phi}_k \simeq -3H^2\Phi_k - \frac{\rho_D}{M_p^2}\Phi_k + \frac{2\rho_D}{M_p^2 r_{h0}}\int_t^{\infty} \frac{dt'}{a(t')}\Phi_k(t') \ . \quad (11)$$

The integral in the last term can be estimated using

$$\int_t^\infty \frac{dt'}{a(t')}\Phi_k(t') \simeq \int_t^\infty \frac{dt'}{a(t')}\left(\Phi_k(t) + \dot{\Phi}_k(t)(t'-t)\right) . \tag{12}$$

Note that for $\int_t^\infty \frac{dt'}{a(t')}(t'-t)$, using $dt'/a(t') = -dr_{h0}$ and expanding $R_{h0}(t') \simeq R_{h0}(t) + (t'-t)\dot{R}_{h0}(t)$, we have

$$\int_t^\infty \frac{dt'}{a(t')}(t'-t) \simeq r_{h0}R_{h0} + \int_t^\infty \frac{dt'}{a(t')}(t'-t)\left(c\sqrt{\frac{\rho_c}{\rho_D}} - 1\right) , \tag{13}$$

where ρ_c is the critical density. So we see that for the dark energy dominated era and $c \approx 1$, the expansion works well. Up to the leading order, $\int_t^\infty \frac{dt'}{a(t')}(t'-t) \simeq r_{h0}R_{h0}$. So Eq. (9) can be written as

$$(2c\sqrt{\frac{\rho_D}{\rho_c}} - 1)\dot{\Phi}_k = H(1 - \frac{\rho_D}{\rho_c})\Phi_k . \tag{14}$$

During the dark energy domination, $1 - \frac{\rho_D}{\rho_c}$ approaches zero quickly, so the super-horizon perturbation approaches a constant. There is no instability for the perturbation.

Note that when $c \simeq 1/2$, there exists a parameter region where $2c\sqrt{\rho_D/\rho_c} - 1$ approaches zero before $\rho_D \to \rho_c$. In this case, the next to leading order correction to (13) should be considered. However, the experimental data indicates that c does not lie in this regime.

For the sub-horizon mode $kr_{h0}(t) \gg 1$, similar analysis can be performed. One can divide the nonlocal integration in (9) into two parts, namely $kr_{h0}(t') \gg 1$ and $kr_{h0}(t') \ll 1$, and use $|\sin[kr_{h0}(t')]| < 1$ and $|\sin[kr_{h0}(t')]| < kr_{h0}(t')$ respectively. It can be shown that the dominant contribution comes from

$$3H\dot{\Phi}_k \simeq -\left(\frac{k^2}{a^2} + 3H^2\right)\Phi_k , \tag{15}$$

and the other terms are suppressed by a factor of $1/[kr_{h0}(t)]$. So the sub-horizon mode is a decaying mode. Again, no instability appears.

Before proceeding to the next section, we would like to discuss two physical issues of the holographic dark energy.

First, from the calculation above, we see clearly that perturbation of holographic dark energy is nonlocal. This is completely different from a usual fluid component. For a usual fluid component, the perturbation equation has a non-vanishing local

limit. In this limit, the perturbation equation follows from local conservation of the energy-momentum tensor $\partial_\mu T^{\mu\nu} = 0$, and the sound speed c_s for the perturbation is given by $c_s^2 = dp/d\rho$. When $c_s^2 = dp/d\rho < 0$, the perturbation of the fluid is unstable. But for holographic dark energy, the perturbation of the energy density comes from the perturbation of the metric, and does not suffer such instability.

Second, although we discussed the evolution of perturbation for holographic dark energy, we did not discuss the initial condition for it. As it is difficult to write the holographic dark energy component into the Lagrangian, the quantum initial condition for holographic dark energy is not available. Another source for perturbation of holographic dark energy is perturbation of the matter component. Perturbation of the matter component couples to the metric perturbation, thus providing a initial condition for the holographic dark energy perturbation.

3 The Fate of Interacting Holographic Dark Energy

In this section we will make a simple and phenomenological classification of interacting holographic dark energy. We also study the fate of the universe with interacting holographic dark energy: in what case it will be phantom-like, in what case phantom will be avoided, and whether the big rip will happen or not.

For simplicity, use w to denote the effective index of the equation of state of dark energy (which is sometimes written as $w_D^{\rm eff}$ in the literature), and use w_m to denote the effective index of the equation of state of matter (which is sometimes written as $w_m^{\rm eff}$ in the literature) in the following discussion. Please note that we neglect the curvature of the universe, the following calculation was done with the assumption that the universe is flat.

3.1 Dark Energy Decay to Matter

First, we consider the case when holographic dark energy decays to matter

$$\rho_D' + 3(1+w_D)\rho_D = 3b\rho_D \ , \tag{16}$$

$$\rho_m' + 3\rho_m = -3b\rho_D \ , \tag{17}$$

where the prime denotes derivative with respect to $\ln a$. It is worthy to note that the total energy is conserved in the interacting holographic dark energy model, although dark energy and matter are not conserved separately. For the lack of the first principle of holographic dark energy, we take the above equation phenomenologically. We simply follow other works done about the interacting holographic dark energy[9]. The decay rate is proportional to the energy density of dark energy, so naturally we have $b < 0$, meaning that dark energy decays to matter. Moreover, $b > 0$ will lead to unphysical consequence in physics, such as ρ_m will become negative and Ω_D will be larger than 1 in the future. So we assume $b < 0$ in this subsection.

Comparing with the effective equation of state

$$\rho_D' + 3(1+w)\rho_D = 0 \ , \quad \rho_m' + 3(1+w_m)\rho_m = 0 \ , \tag{18}$$

we find the indices of the effective equation of state

$$w = w_D - b \ , \quad w_m = b\Omega_D/\Omega_m \ . \tag{19}$$

If the index of the effective equation of state of dark energy satisfies $w < -1$, dark energy is phantom-like.

Taking derivative of Eq.(1) with respect to $\ln a$, we have

$$\rho_D' = 2\rho_D \left(\frac{\sqrt{\Omega_D}}{c} - 1 \right) \ , \tag{20}$$

from Eqs.(16)(20), we get

$$w_D = -\frac{1}{3} - \frac{2}{3}\frac{\sqrt{\Omega_D}}{c} + b \ . \tag{21}$$

Using the definition of Ω_D and taking derivative of Ω_D with respect to $\ln a$, we have

$$\Omega_D' = -2\Omega_D + \frac{2\Omega_D^{3/2}}{c} - 2\Omega_D\frac{H'}{H} \ . \tag{22}$$

From Eq.(22), we get

$$\frac{H'}{H} = -\frac{\Omega_D'}{2\Omega_D} - 1 + \frac{\sqrt{\Omega_D}}{c} \ . \tag{23}$$

From the Friedmann equation

$$\dot{H} = -4\pi G(\rho + p) = -4\pi G(\rho + \rho_D w_D + \rho_r w_r) \ , \tag{24}$$

we get

$$\frac{H'}{H} = \frac{\Omega_D}{2} + \frac{\Omega_D^{3/2}}{c} - \frac{3}{2}b\Omega_D - \frac{3}{2} \ , \tag{25}$$

the last term of the RHS of Eq.(24) can be neglected.

Substituting Eq.(25) into Eq.(22), we obtain the differential equation for Ω_D

$$\frac{\Omega_D'}{\Omega_D} = (1-\Omega_D)(1+\frac{2\sqrt{\Omega_D}}{c}) + 3b\Omega_D \ . \tag{26}$$

We solve Eqs.(25)(26) numerically and the evolution of the universe has been shown in Fig. 3. Now we will discuss these equations analytically.

Considering Eq.(26), we find that the LHS of Eq.(26) will vanish only when the scale factor goes to infinity. To see this, we define

$$f(y) \equiv \frac{2y'}{y} = (1-y^2)(1+\frac{2y}{c}) + 3by^2 \ , \tag{27}$$

where $y \equiv \sqrt{\Omega_D}$. The equation $f(y) = 0$ has three roots. Since $f(0) = 1 > 0$ and $f(1) = 3b < 0$, there is one root in the region [0,1] at least. We only consider the region $[0, 1]$ since it is the physical region, for Friedmann equation in the flat universe, the energy density of holographic dark energy and that of matter should be positive. So Ω_D should never go beyond the region [0,1]. We assume that y_1 is the first root in the physical region $[0, 1]$. Thus, we find the integral equation

$$\int_{\varepsilon}^{y_1} \frac{dy}{y\prod_{i=1}^{3}(y-y_i)} = \ln a - \ln a(\varepsilon) \ , \tag{28}$$

where ε is a cut-off at the early universe. We find that the LHS of Eq.(28) diverges, which means that the scale factor approaches infinity as $y \to y_1$. Combining the above equation with the fact that $\Omega_D' > 0$ when $\Omega_D \to 0$, we conclude that Ω_D' will remain positive, and approaches zero when the scale factor approaches infinity.

When the index of the effective equation of state satisfies $w = w_D - b \geq -1$, *i.e.* $\frac{\sqrt{\Omega_D}}{c} \leq 1$, dark energy does not behave like phantom. As we have discussed, Ω_D is an increasing function with the scale factor. So once the no phantom condition $\frac{\sqrt{\Omega_D}}{c} \leq 1$ is satisfied at $a \to \infty$, it will be satisfied along the whole history of the universe. By setting $\Omega_D' = 0$ in Eq.(26), we get $b \leq 1 - c^{-2}$. It is the necessary condition to avoid the phantom phase.

We can also prove that $b \leq 1 - c^{-2}$ is the sufficient condition of no phantom. To see this, we first investigate the limiting case $b = 1 - c^{-2}$, then use the monotonicity

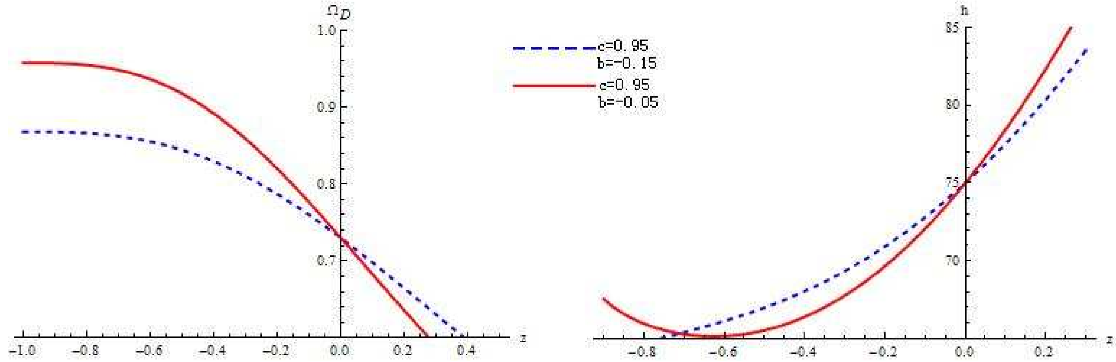

Figure 3: This figure illustrates evolution of the universe with assumption that holographic dark energy can decay to matter. The red (solid) curve corresponds to the phantom-like universe, and blue (dashed) curve corresponds to the other case. We choose initial condition $\Omega_{D0} = 0.73$, $h_0 = 75$ in our numerical calculation.

of Ω_D in b to prove for the general case. Substituting $b = 1 - c^{-2}$ into $f(y) = 0$, we get three roots, $y_1 = c$, $y_{2,3} = \frac{-3\pm\sqrt{9-8c^2}}{4c}$. Note that $y_{2,3} < 0$, so y_1 is the only root making sense physically. Note that solution $y_1 = c$ corresponds to $w = -1$, which is the boundary of the no phantom condition.

To see the monotonicity, suppose y_i is the root of $f(y) = 0$,

$$(1 - y_i^2)(1 + \frac{2y_i}{c}) + 3by_i^2 = 0\ . \tag{29}$$

Taking derivative of the above equation with respect to b, we get,

$$\frac{dy_i}{db} = \frac{-3y_i^2}{-2y_i + \frac{2}{c} - \frac{6y_i^2}{c} + 6by_i}\ . \tag{30}$$

There is only one inflexion, $\frac{dy_i}{db} = 0$, $y_i = 0$, so the function varies monotonously when $y_i \geq 0$. Substitute $b = 1 - c^{-2}$ into Eq.(30), one finds $\frac{dy_i}{db} > 0$, around $y_i \to c$. That means $y = c$ is largest root of Eq.(29). So we conclude that for general b satisfying $b \leq 1 - c^{-2}$, we have $\frac{\sqrt{\Omega_D}}{c} \leq 1$, and phantom will be avoided. Thus, we conclude that $b \leq 1 - c^{-2}$ is the sufficient and necessary condition of no phantom.

In dark energy models, phantom usually causes the big rip. In the remainder of this subsection, we will verify that this statement is also true for the interacting holographic dark energy. First, we will prove that if the no phantom condition $b \leq 1 - c^{-2}$ is satisfied, there will be no big rip.

From equation

$$\dot{H} = -4\pi G[(1 + w)\rho_D + (1 + w_m)\rho_m], \tag{31}$$

where $w_m = \frac{b\Omega_D}{\Omega_m} \geq -\frac{1}{3}(1+\frac{2\sqrt{\Omega_D}}{c}) \geq -1$, we get $\dot{H} \leq 0$, the Hubble parameter will become smaller and smaller and big rip will never happen.

On the other hand, if dark energy is phantom-like, we can show that the big rip will definitely happen. Consider the asymptotic behavior of the evolution equation. When $a \to \infty$, to have phantom, we have $1 + w = 1 + w_m = -\alpha$, where α is a positive constant. Eq.(31) can be rewritten as, $\dot{H} = -4\pi G[-\alpha(\rho_D + \rho_m)] = \frac{3\alpha}{2}H^2$ in the $a \to \infty$ limit, and $H = \frac{1}{H_0 - 3\alpha t/2}$, where H_0 is a integral constant. So the big rip happens in a finite time $t = \frac{3}{2}\frac{H_0}{\alpha}$.

3.2 Hybrid Interaction

For the hybrid interaction, the interacting term is proportional to the critical energy density. The evolution equations are

$$\rho_D' + 3(1 + w_D)\rho_D = 3b\rho_c \ , \tag{32}$$

$$\rho_m' + 3\rho_m = -3b\rho_c \ . \tag{33}$$

Dark energy will become dominant in this case, and $\rho_D \to \rho_c$. For the same reason as in Subsection 3.1, we consider the case $b < 0$.

With a same procedure as in the previous subsection, we obtain differential equations for Ω_D and H

$$\frac{\Omega_D'}{\Omega_D} = (1 - \Omega_D)(1 + \frac{2\sqrt{\Omega_D}}{c}) + 3b \ , \tag{34}$$

$$\frac{H'}{H} = \frac{\Omega_D}{2} + \frac{\Omega_D^{3/2}}{c} - \frac{3}{2}b - \frac{3}{2} \ . \tag{35}$$

Solving above two equations numerically, We get the evolution of the universe, which has been shown in Fig. 4. The indices of the effective equations of state of holographic dark energy and matter are

$$w = -\frac{1}{3} - \frac{2\sqrt{\Omega_D}}{3c} \ , \quad w_m = \frac{b}{\Omega_m} \ . \tag{36}$$

Using Eq.(34) and the condition $w = -\frac{1}{3} - \frac{2\sqrt{\Omega_D}}{3c} \geq -1$, we get the sufficient condition of no phantom is $b \leq c^2 - 1$. This condition is also the necessary condition, as can be shown in the same way as in the previous subsection. We rewrite Eq.(34) in terms of $y \equiv \sqrt{\Omega_D}$,

$$f(y) \equiv \frac{2y'}{y} = (1 - y^2)(1 + \frac{2y}{c}) + 3b \ . \tag{37}$$

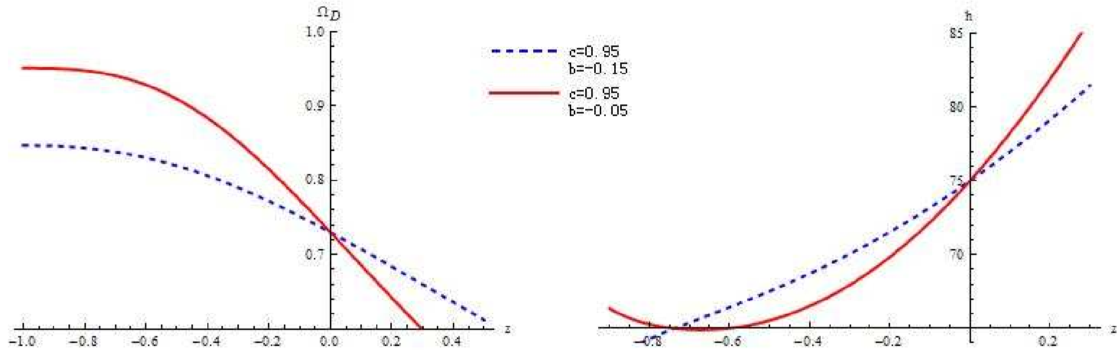

Figure 4: This figure illustrates evolution of the universe with assumption that the interacting term is proportional to the critical energy density. The red (solid) curve corresponds to the phantom-like universe, and blue (dashed) curve corresponds to the other case. We find that evolution is very similar with the case dark energy decay to matter.

Assume that y_i is the root of $f(y) = 0$,

$$(1 - y_i^2)(1 + \frac{2y_i}{c}) + 3b = 0 \ . \tag{38}$$

There are three roots in the limit case $b = c^2 - 1$,$y_1 = c$, $y_{2,3} = \frac{-3c \pm \sqrt{16-15c^2}}{4}$.

In addition, Eq.(28) implies that there is a constraint on b. From Eq.(28), we know that holographic dark energy evolves monotonously along the whole history of the universe. Take into account the fact that Ω_D is close to zero at the beginning of evolution of the universe, Ω_D' must be positive all the time. To meet this requirement, we need $b > -1/3$. Otherwise, Ω_D will be a decreasing function in time. Combining this with Eq.(38), we obtain $\frac{2}{c} < \frac{2y_i}{c} + y_i$.

Taking derivative of Eq.(38) with respect to b, we get

$$\frac{dy_i}{db} = \frac{-3}{-\frac{6y_i^2}{c} - 2y_i + \frac{2}{c}} \ . \tag{39}$$

We see that the denominator is negative since $\frac{2}{c} < \frac{2y_i}{c} + y_i$. So y_i increases monotonously with respect to b in the physical region. If we tune b such that $b \leq c^2 - 1$, we have $w = -\frac{1}{3} - \frac{2}{3}\frac{\sqrt{\Omega_D}}{c} \leq -1$ and phantom phase will be avoided.

By a similar analysis as in the previous subsection, we find that $b \leq c^2 - 1$ is also the sufficient and necessary condition of no big rip.

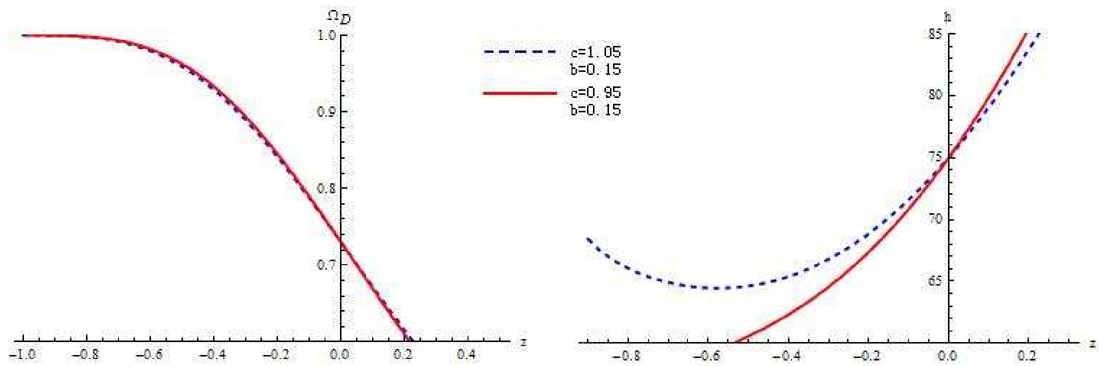

Figure 5: This figure illustrates evolution of the universe with assumption that matter can decay to dark energy. The red (solid) curve corresponds to the phantom-like universe, and blue (dashed) curve corresponds to the other case. We find that dark energy will be dominant and $\Omega_D = 1$ eventually.

3.3 Matter Decay to Dark Energy

Finally, we consider the case when matter decays to dark energy,

$$\rho_D' + 3(1+w_D)\rho_D = 3b\rho_m \ , \tag{40}$$

$$\rho_m' + 3\rho_m = -3b\rho_m \ . \tag{41}$$

In this case, $b \geq 0$, and the differential equation for Ω_D takes the form

$$\frac{\Omega_D'}{\Omega_D} = (1-\Omega_D)(1+\frac{2\sqrt{\Omega_D}}{c}) + 3b(1-\Omega_D) \ . \tag{42}$$

The differential equation for H can be written as

$$\frac{H'}{H} = \frac{\Omega_D}{2} + \frac{\Omega_D^{3/2}}{c} - \frac{3b}{2}(1-\Omega_D) - \frac{3}{2} \ . \tag{43}$$

We obtain the indices of the effective equation of state

$$w = -\frac{1}{3} - \frac{2\sqrt{\Omega_D}}{3c} \ , \quad w_m = b \ . \tag{44}$$

Taking Ω_D' in Eq.(42), there is only one root in the range [0,1], dark energy will be dominant, and Ω_D approaches 1 eventually. The sufficient and necessary condition of no phantom is $c \geq 1$. It is also the sufficient and necessary condition of no big rip. The analysis is just a little different from the previous ones, because $w_m = b$ all the

time and there is no attractor solution to the evolution equation of state. Consider equation,

$$\dot{H} = -4\pi G[(1+w)\rho_D + (1+b)\rho_m], \tag{45}$$

If no phantom, *i.e.* $1+w \geq 0$, $\dot{H} < 0$ all the time, and the big rip will never happen. If dark energy is phantom-like, $c < 1$, Note $\Omega_D \to 1$ eventually, we can neglect the second term in Eq.(45), $H = \frac{1}{H_0-(c^{-1}-1)t}$ when $t = \frac{H_0}{c^{-1}-1}$, the big rip will occur. The evolution of the universe in this case has been shown in Fig. 5.

4 The Coincidence Problem

The usual solution to the coincidence problem is to calculate the ratio of duration of coincidence state and lifetime of the universe, assuming that dark energy is phantom-like and the universe will end with the big rip [10]. The coincidence problem is solvable if this ratio is not too small. Another solution to the coincidence problem is the interacting dark energy [9]. We shall show that the interacting dark energy does not solve the coincidence problem in this section. We consider the solution proposed by Li in the original paper of holographic dark energy [6] a more natural solution.

We start with evolution equations

$$\rho_D' + 3(1+w)\rho_D = 0 \ , \tag{46}$$

$$\rho_m' + 3(1+w_m)\rho_m = 0 \ , \tag{47}$$

where the prime denotes derivative with respect to $\ln a$, and w, w_m are the effective indices of the equations of state of holographic dark energy and matter, respectively.

We rewrite Eq.(46) in terms of an integral form

$$\ln \rho_D/\rho_{D0} = \int_0^{\ln a} -3(1+w) d\ln a' \tag{48}$$

Recall the median law of integral, we write the median value of w as a constant $\tilde{w}$ varying in the interval $(-\frac{1}{3}, -\frac{1}{3} - \frac{2}{3c})$. So the above integration can be written as $\ln \rho_D/\rho_{D0} = -3(1+\tilde{w})\ln a$, we get $\rho_D = \rho_{D0} a^{-3(1+\tilde{w})}$. Similarly we write the median value of w_m as a constant $\tilde{w}_m$, and we get $\rho_m = \rho_{m0} a^{-3(1+\tilde{w}_m)}$, where ρ_{D0} and ρ_{m0} are energy densities at the time where we set the scale factor $a_0 = 1$.

From the Friedmann equation

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_D) \ , \tag{49}$$

and the definition of the Hubble parameter $H = \frac{d \ln a}{dt}$, the age of the universe can be written in an integral form as follows

$$t = \int_0^{a_0} [\frac{8\pi G}{3}(\rho_m + \rho_D)]^{-1/2} d \ln a \ . \tag{50}$$

Rewrite it in terms of $r \equiv \frac{\rho_D}{\rho_m}$, we obtain

$$t = H_0^{-1}(1+r_0)^{1/2} r_0^{\frac{1+\tilde{w}}{2(\tilde{w}-\tilde{w}_m)}} \frac{1}{-3(\tilde{w}-\tilde{w}_m)} \int_0^{r_0} (1+r)^{-1/2} r^{-\frac{1+\tilde{w}_m}{2(\tilde{w}-\tilde{w}_m)}-1} dr \ , \tag{51}$$

The integration in Eq.(51) satisfies the follow inequality

$$\int_0^{r_0} (1+r_0)^{-1/2} r^{-\frac{1+\tilde{w}_m}{2(\tilde{w}-\tilde{w}_m)}-1} dr < \int_0^{r_0} (1+r)^{-1/2} r^{-\frac{1+\tilde{w}_m}{2(\tilde{w}-\tilde{w}_m)}-1} dr \ , \tag{52}$$

and

$$\int_0^{r_0} (1+r)^{-1/2} r^{-\frac{1+\tilde{w}_m}{2(\tilde{w}-\tilde{w}_m)}-1} dr < \int_0^{r_0} r^{-\frac{1+\tilde{w}_m}{2(\tilde{w}-\tilde{w}_m)}-1} \ . \tag{53}$$

So we get

$$\frac{2}{3} H_0^{-1}(1+\tilde{w}_m)^{-1} < t < \frac{2}{3} H_0^{-1}(1+r_0)^{1/2}(1+\tilde{w}_m)^{-1} \ . \tag{54}$$

If we input the value of r_0 at the present time [3][4], we find that t is about $H_0^{-1}(1+\tilde{w}_m)^{-1}$. From the previous section, we know that w_m is proportional to b, the interacting parameter. We see the coefficient of H_0^{-1} is b dependent. For $b = 0$, t is about H_0^{-1}. For $b \neq 0$, from Eq.(19), $w_m = \frac{b\Omega_D}{\Omega_m}$ for dark energy decaying to matter case, and from Eq.(36), $w_m = \frac{b}{\Omega_m}$ for hybrid interaction case, where b is always negative. From Eq.(44), we have $w_m = b$ $(b > 0)$ for the case in which matter decaying to dark energy.

The key issue of the coincidence problem is why the ratio of holographic dark energy to matter is order one nowadays, in other words, why the ratio is order one when the age of universe is about 10^{10} years. The order of magnitude of the age of the universe is determined by H_0^{-1}. The relationship between the age of the universe and H_0^{-1} depends on the initial condition of the universe. The interacting term can change the coefficient in the front of H_0^{-1}, but it can not provide any information about the value of H_0^{-1}. Thus, the interacting holographic dark energy can not solve

the coincidence problem completely in this sense. The above approach just solves the coincidence problem partially.

To solve the coincidence problem completely, the initial density of holographic dark energy and the influence of inflation should be taken into account. As proposed in the original paper of holographic dark energy [6], the initial energy density of holographic dark energy has been inflated away by a factor $\exp(-2N)$ in the inflation epoch, where N is the e-folding number of inflation. So the ratio between ρ_D and ρ_r, the radiation density, should be about 10^{-52} at the onset of the radiation dominated epoch, if we suppose that the inflation energy scale be 10^{14} GeV and inflaton energy completely decays into radiation at the end of inflation. This will lead to the order 1 ratio of holographic dark energy to matter in our epoch. Thus inflation not only solves the traditional naturalness problems and helps to generate primordial perturbations, but also solves the cosmic coincidence problem.

5 Conclusion and Discussion

In this paper, we study perturbation of holographic dark energy. Since holographic dark energy is just the holographic vacuum energy, its perturbation is global. We calculate perturbation of the holographic dark energy, and find it stable.

Many numerical and analytic works have been done on the interacting holographic dark energy. We made a simple and phenomenological classification of interacting holographic dark energy in this paper, and derived the sufficient and necessary condition of no phantom (the big rip). Needless to say, this classification has not been done previously. It is worth to note that we write the interacting term just by hand for lack of knowledge of the first principle of holographic dark energy. We hope we will return to this issue in future projects.

We also discussed the coincidence problem. It is shown that the interacting holographic dark energy approach only solves the coincidence problem partially. The original solution to the coincidence problem proposed in [6] stands a better resolution.

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